Resource Equivalences in Petri Nets

Irina A. Lomazova

National Research University Higher School of Economics,
Moscow, Russia

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Outline

Petri Nets: a Resource Perspective

In Petri nets tokens often represent different kinds of resources.

Wide use of Petri nets for modeling resource-oriented systems:

- manufacturing systems,
- resource allocation systems,
- etc.

Special resource places in

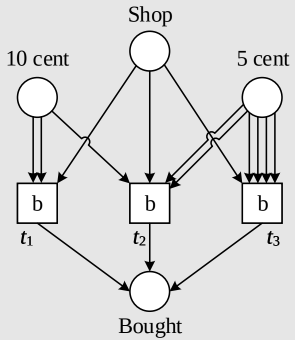
- *open nets* [R. Heckel 2003, P. Baldan et al 2009, X. Dong et al 2016, etc.], and
- *workflow nets* [N. Sidorova, and C. Stahl 2013, V. Bashkin, and I. Lomazova 2014, etc.].

Close connection with the

- *Linear logic* of J.Y. Girard studied by B. Farwer (Muller).

Introduction: Tokens as Resources

Buying goods for 20 cents



Three initial markings

- **10 cent** + **10 cent**,
- **10 cent** + **5 cent** + **5 cent**,
- **5 cent** + **5 cent** + **5 cent** + **5 cent**

generate the same behavior.

Moreover, **10 cent** coin is equivalent to two **5 cent** coins in any state.

Challenge:

to find equivalent resources for a given Petri net.

Why?

May be helpful for:

- state space reduction;
- Petri net reduction;
- adaptive process control;
- process optimization;
- ...

The talk gives an overview of some results of joint work with *Vladimir Bashkin*, Yaroslavl State University, Russia

Petri Nets: Notation

$\mathcal{M}(S)$ denotes the set of all finite multisets over S.
For $m, m' \in \mathcal{M}(S)$

- $m \subseteq m'$ iff $\forall s \in S : m(s) \leq m'(s)$, and
- $\forall s \in S : m+m'(s) = m(s) + m'(s)$.

Labeled Petri net is

a tuple $N = (P, T, W, l)$, where

- P — a set of *places*,
- T — a set of *transitions*,
- $W : (P \times T) \cup (T \times P) \to \mathbb{N}$ — an arc-weight function,
- $l : T \to Act$ — a labeling function.

$M \in \mathcal{M}(P)$ denotes a marking in a Petri net,
$M \xrightarrow{t} M'$ — a transition firing.

Marking Bisimulation

Informally: Two markings are considered equivalent, if they generate the same observable behavior.

$$\begin{array}{ccc} M_1 & R & M_2 \\ \downarrow t & & \downarrow u \\ M_1' & R & M_2' \end{array}$$

$$l(u) = l(t)$$

A relation $R \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ satisfies the *transfer property* iff for all $(M_1, M_2) \in R$ and for every firing $M_1 \xrightarrow{t} M_1'$, there exists an imitating firing $M_2 \xrightarrow{u} M_2'$, such that $l(t) = l(u)$, and $(M_1', M_2') \in R$.

Definition

A relation R is called a *marking bisimulation*, if both R and R^{-1} satisfy the transfer property (denoted $\sim$).

Marking Bisimulation

Finding equivalent states is very helpful for reducing complexity of Petri nets analysis.

For two bisimular markings replacing one of them by another does not change the observable system behavior.

B U T

Marking bisimulation *is undecidable* for Petri nets [P. Jančar, 1994]

Place Fusion

Place fusion [Ph. Schnoebelen and N. Sidorova, 2000]

an equivalence relation on Petri net places:

a token in one place can be replaced by a token in the other one in any marking without changing the net observable behavior

$$\Longrightarrow$$

merging two equivalent places gives the net with the same observable behavior.

BAD NEWS

Place fusion is undecidable [Ph. Schnoebelen and N. Sidorova, 2000]

Resource Similarity

A *resource* is a 'submarking'.

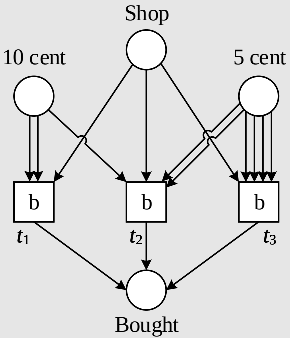
Two resources are *similar* iff replacing one of them by another in *any* reachable marking does not change the observable net behavior.

Example:

the resources **10 cent** and **5 cent** + **5 cent** are similar

10 cent $\approx$ **5 cent** + **5 cent**

Buying goods for 20 cents



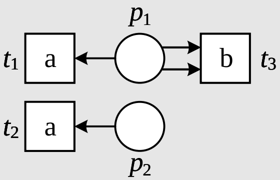
Resource Similarity

Definition

Let r, s be resources in a Petri net $N = (P, T, W, l)$.
$r \approx s$ iff for every $M \in \mathcal{M}(P)$, $r \subseteq M$ implies $M \sim M - r + s$.

Resource similarity is an equivalence, and is more strong than the marking bisimulation:

$$m \approx m' \Rightarrow m \sim m'$$



$p_1 \sim p_2$, but $p_1 \not\approx p_2$

BAD NEWS

Resource similarity is undecidable,
since place fusion is a resource similarity for 1-token resources.

Resource Similarity

GOOD NEWS

Resource similarity has a finite representation.

- Resource similarity is closed under transitivity.
- Resource similarity is closed under addition of resources:

$$r \approx s \;\&\; u \approx v \;\Rightarrow\; r + u \approx s + v$$

- Resource similarity has a finite AT-basis, i.e. for each Petri net N there exists a finite relation B, s.t. its additive-transitive closure B^{AT} coincides with the resource similarity for N.

Well-Quasi-Orderings on Petri Net Markings

Definition

A *well-quasi-ordering* (a **wqo**) is a quasi-ordering $\leq$, such that for any infinite sequence $x_0, x_1, \ldots$ in X, there exist indexes $i < j$ with $x_i \leq x_j$.

In Petri net theory wqo's play an important role:

- The set of minimal w.r.t. a wqo elements is finite.
- By Higman's lemma (1952) sequence (coordinate-wise) partial ordering on the set of a Petri net markings is a wqo.
- The classical coverability tree by Karp and Miller (1969) are based on this wqo and the monotonicity of transition firings.
- In the more general setting this leads to the theory of well-structured transition systems (WSTS) by A. Finkel et al.
- The theory of WSTS was a source for results on decidability for extensions of Petri nets – nested Petri nets [I. Lomazova 2000] and Petri nets with data [F. Rosa-Velardo, et al 2011, S. Lasota 2016].

Ordering on Pairs of Resources

Define a partial order $\sqsubseteq$ on the set $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ of pairs of resources as follows:

1. For identity pairs let

$$(r_1, r_1) \sqsubseteq (r_2, r_2) \overset{def}{\Leftrightarrow} r_1 \subseteq r_2;$$

2. For two non-identity pairs, the maximal identity parts and the addend pairs of disjoint resources are compared separately:

$$(r_1 + o_1, r_1 + o_1') \sqsubseteq (r_2 + o_2, r_2 + o_2') \overset{def}{\Leftrightarrow}$$

$$\overset{def}{\Leftrightarrow} o_1 \cap o_1' = \emptyset \,\&\, o_2 \cap o_2' = \emptyset \,\&\, r_1 \subseteq r_2 \,\&\, o_1 \subseteq o_2 \,\&\, o_1' \subseteq o_2'.$$

3. An identity pair and a non-identity pair are always incomparable.

Resource Similarity is Finitely-Based

Given a relation B, let B_s denote the set of all minimal with respect to $\sqsubseteq$ elements of B^{AT}.

Theorem 1

Let $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ be a symmetric and reflexive relation. Then B_s is an AT-basis of B^{AT} and B_s is finite. [a]

[a] A similar result (in terms of congruences in commutative semigroups) was obtained by L. Redei in 1965 and Y. Hirshfeld (a shorter proof) in 1994. Our proof is based on the explicitly defined wqo, i.e. it is constructive.

Resource similarity is symmetric and reflexive, and hence it can be represented by a finite number of pairs.

B U T,

its finite basis cannot be computed effectively.

Resource Bisimulation

Resource similarity narrows marking bisimilarity, but is still undecidable. ☹

Looking for a computable approximation of the marking bisimilarity...

Definition

An equivalence $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is called a *resource bisimulation* iff B^{AT} is a marking bisimulation.

Resource Bisimulation: Properties

Some immediate properties:

Let N be a labeled Petri net.
Then

1. if B_1, B_2 are resource bisimulations for N then $B_1 \cup B_2$ is a resource bisimulation for N;
2. there exists the largest resource bisimulation, denoted by $\simeq$, such that for every resource bisimulation B we have $B \subseteq (\simeq)$.
3. for a given Petri net the resource bisimilarity is a narrowing of the resource similarity, i.e. for each two resources r and s:

$$r \simeq s \Rightarrow r \approx s.$$

Resource Bisimulation is Finitely-Based

The relation $\simeq$ is symmetric, reflexive, and closed under transitivity and addition of resources $\Longrightarrow$ by Theorem 1 it has a *finite ground AT-basis*, consisting of minimal (w.r.t. wqo $\sqsubseteq$) pairs.

Weak Transfer Property

Note, that AT-closure of a resource bisimulation is a marking bisimulation.

Weak variant of the transfer property (defined in [C. Autant, Ph. Schnoebelen 1992] for place bisimulation)

A relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ satisfies *the weak transfer property* iff for $(r, s) \in B$, $t \in T$, s.t. ${}^\bullet t \cap r \neq \emptyset$, there exists an imitating step $u \in T$, s.t. $l(t) = l(u)$, and ${}^\bullet t \cup r \xrightarrow{t} M_1$, ${}^\bullet t - r + s \xrightarrow{u} M_2$, where $(M_1, M_2) \in B^{AT}$.

$$r \quad (B) \quad s$$

$$\begin{array}{ccc} {}^\bullet t \cup r & & {}^\bullet t - r + s \\ \downarrow t & & \downarrow (\exists) u \\ M_1 & (B^{AT}) & M_2 \end{array}$$

$$l(u) = l(t)$$

Resource Bisimulation: Characterization

Theorem 2

A relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is a resource bisimulation iff B is an equivalence and it satisfies the weak transfer property.

GOOD NEWS

Weak transfer property can be directly verified for a given finite relation ☺

Computing the Largest Resource Bisimulation for Bounded Resources: **Algorithm**

Input: a labeled Petri net $N = (P, T, F, l)$, a positive integer q.
Output: the relation $B(N, q)$ on the set of resources with not more than q tokens.

Step 1: Let $C = \{(\emptyset, \emptyset)\}$.
Step 2: Let $B = (\mathcal{M}_q(P) \times \mathcal{M}_q(P)) \setminus C$.
Step 3: Compute a ground basis B_s. Denote by B_s^{nr} the set of all non-identity elements of B_s.
Step 4: Check if B_s conforms to the weak transfer property (it is sufficient to check only B_s^{nr}). If YES, then stop. The current B is $B(N, q)$. Otherwise, find $(r, s) \in B_s^{nr}$ and a transition $t \in T$ with ${}^\bullet t \cap r \neq \emptyset$ such that ${}^\bullet t \cup r \xrightarrow{t} {M_1}'$ cannot be imitated from ${}^\bullet t - r + s$. Then add pairs (r, s), (s, r) to C and return to **Step 2**.

Resource Bisimulation for Bounded Resources

$B(N, q)$ is a finite approximation for $\simeq$.

$$B(N,1) \subseteq B(N,2) \subseteq B(N,2) \subseteq \cdots \subseteq \simeq,$$

and hence

$$B(N,1)^{AT} \subseteq B(N,2)^{AT} \subseteq B(N,3)^{AT} \subseteq \cdots \subseteq \simeq .$$

But since the resource bisimilarity has a finite AT-basis, there exists a natural number q_0 such that

$$B(N,1)^{AT} \subseteq \cdots \subseteq B(N,q_0)^{AT} = B(N,q_0{+}1) = \cdots = \simeq .$$

Resource Bisimulation: Open Problem 1

Computing an upper bound for q_0 is equivalent to computing an AT-basis for the resource bisimilarity.

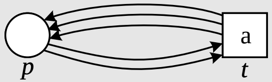
OPEN PROBLEM

Can the largest resource bisimulation be effectively computed?

Resource Bisimulation: Open Problem 1

$B(N, q) = B(N, q+1)$ for some q does not imply that the sequence has stabilized.

For this net we have



$B(N, 1) = B(N, 2) = \{(0, 0), (1, 1)\}$,

$B(N, 3) = \{(0, 0), (1, 1), (2, 3), (3, 2)\}$, and

$B(N, 1)^{AT} = B(N, 2)^{AT} \subset B(N, 3)^{AT} = \simeq$.

Resource Bisimulation: Open Problem 2

The largest resource bisimulation $\simeq$ and resource similarity $\approx$

WE KNOW

$$(\simeq) \subseteq (\approx)$$

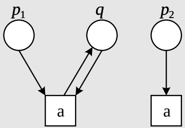
OPEN PROBLEM

$$(\simeq) \stackrel{?}{=} (\approx)$$

Conditional Resource Similarity

Resources r and s are called *similar under a condition* b (denoted $r \approx_{|b} s$) iff for every resource $m \in \mathcal{M}(P)$ s.t. $b \subseteq m$ we have $m + r \sim m + s$.

Resources r and s are called *conditionally similar* (denoted $r \approx_{|} s$) iff there exists $b \in \mathcal{M}(P)$ s.t. $r \approx_{|b} s$.



$p_1 \approx_{|q} p_2$, $p_1 \not\approx p_2$

$p \approx_{|p} \emptyset$

Conditional Resource Similarity: Properties

- Conditional similarity is an equivalence.
- Plain and conditional similarities:
  For $m \approx m'$ we have $m + r \approx m' + s$ iff $r \approx_{|m} s$
- Conditional similarity is closed under addition of resources:
  If $m \approx_{|b_1} m'$, and $r \approx_{|b_2} r'$, then $m + r \approx_{|b_1 \cup b_2} m' + r'$.
- Monotonicity of conditions:
  $r \approx_{|b} s,\ b \subseteq b' \ \Rightarrow\ r \approx_{|b'} s$.
- Common parts can be removed from both similar resources:
  $m + r \approx_{|b} m + s \ \Leftrightarrow\ r \approx_{|b+m} s$.
  $m + r \approx_{|} m + s \ \Leftrightarrow\ r \approx_{|} s$.

Conditional Resource Similarity is Finitely-Based

- Conditional similarity (like the plain similarity) is *closed under addition* of resources $\Longrightarrow$ it has a finite AT-basis.
- (Unlike the plain similarity) conditional similarity is *closed under the subtraction* of similar resources:
  $m \approx m', \ m + r \approx m' + s \ \Rightarrow \ r \approx_| s.$
  $m \approx_| m', \ m + r \approx_| m' + s \ \Rightarrow \ r \approx_| s.$
  $\Longrightarrow$ it has a *finite additive basis*, i.e. every pair of conditionally similar resources can be decomposed into a sum of minimal (indecomposable) pairs of conditionally similar resources.
- The set of conditions for a pair of conditionally similar resources is upward closed. $\Longrightarrow$ for every pair $r \approx_| s$ *the set of all its minimal conditions is finite*.

Plain and Conditional Resource Similarities

Here we say that a pair $r \approx s$ of similar resources is *minimal* iff it cannot be decomposed into a sum of a pair of similar resources and a pair of conditionally similar resources, i.e.
$r' \approx s'$, and $r = r' + r''$, $s = s' + s''$ implies $r = r'$, and $s = s'$.

Then every pair of similar resources can be decomposed into the sum of one minimal pair of similar resources and several minimal pairs of conditionally similar resources.

Decomposing Resource Similarity: Notation

Let $R \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ be some set of pairs of conditionally similar resources, i.e. $R \subseteq (\approx_|)$.
Let $B = \{\, (u, v) \in \mathcal{M}(P) \times \mathcal{M}(P) \mid u \approx v \wedge \forall\, (r, s) \in R : u + r \approx v + s\}$ be a set of all common conditions for R.

By $Cond(R)$ we denote the set of all minimal elements of B (w.r.t. $\leq$, considering B as a set of vectors of length $2|P|$).

Decomposing Resource Similarity

For $R, S \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$,
$lc(R)$ denotes the set of all *linear combinations* over R, i.e.
$lc(R) = \{(r,s) \mid (r,s) = (r_1, s_1) + \ldots + (r_k, s_k), (r_i, s_i) \in R, i = 1, \ldots, k\}$;
and $R + S$ denotes the set of all *sums of pairs* from R and S, i.e.
$R + S = \{(u,v) \mid (u,v) = (r+r', s+s'), (r,s) \in R, (r', s') \in S\}$.

Theorem 3

Let N be a Petri net, $(\approx)$ – the set of all pairs of similar resources for N, $(\approx_|)$ – the set of all pairs of conditionally similar resources for N. The set $(\approx)$ is semilinear. Specifically, there exists a finite set $R \subseteq (\approx_|)$ s.t.

$$(\approx) = \bigcup_{\mathcal{R} \in 2^R} \left[\, Cond(\mathcal{R}) + lc(\mathcal{R}) \,\right]$$

Conditional Resource Similarity: Summary

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- Conditional resource similarity is undecidable ☹,
  since resource similarity is undecidable.
- To find conditionally similar resources one can
  - compute finite resource bisimulations,
  - subtract similar (or identity) parts in pairs of similar resources.
- Since the conditional resource similarity has a finite additive basis, we may hope to chose the right size of the finite bisimilarity relation to compute the additive basis of the conditional resource bisimilarity.

Generalized Resource Similarity

Interchangeable resources may be used for adaptive controlling of the process execution.

Challenge: to replace some resources together with changing the next activity step without changing the observable net behavior.

Generalized resource

consists of

- a (passive) *material resource* — a bag of tokens,
- a (dynamic) *activity resource* — a bag of activities.

Activity resource should be ensured by the necessary material resources.

Here we consider parallel (simultaneous) transition firings.

Generalized Resource Similarity: Definition

Definition

Let $N = (P, T, F, l)$ be a labeled Petri net. A pair (r, α) s. t. $r \in \mathcal{M}(P), \alpha \in \mathcal{M}(T)$ and ${}^\bullet\alpha \subseteq r$ is called a *generalized resource* of a Petri net N.

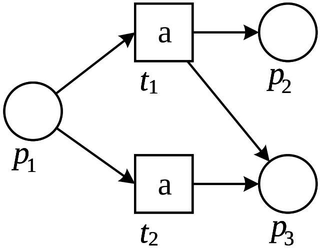
Definition

Generalized resources (r, α) and (s, β) are called *similar* iff

1. $l(\alpha) = l(\beta)$;
2. for every marking $M \in \mathcal{M}(P)$, if $M + r \xrightarrow{\alpha} M'$ and $M + s \xrightarrow{\beta} M''$, then $M' \sim M''$.

By abuse of notation we use $\approx$ for the similarity of generalized resources.

Generalized Resource Similarity: Example



$$(p_1, t_1) \approx (p_1 + p_2, t_2)$$

Generalized Resource Similarity: Expressiveness

- $(r, \alpha) \approx (r, \beta)$ – activities α and β are **equivalent** in any state of the system, when a resource r is available.
- $(r, \alpha) \approx (s, \alpha)$ – resources r and s are **equivalent**, if α is certainly executed.
- $(r, \alpha) \approx (r + s, \beta)$ – activity α is more **efficient** than β.
- $(r, \emptyset) \approx (r + s, \emptyset)$ – resource s is **redundant.**

Generalized Resource Similarity: Properties

- The generalized resource similarity is an equivalence.
- for each two steps $\alpha, \beta \in \mathcal{M}(T)$:

$$I(\alpha) = I(\beta) \quad \Rightarrow \quad ({}^\bullet\alpha + \beta^\bullet, \alpha) \approx ({}^\bullet\beta + \alpha^\bullet, \beta).$$

- A generalized resource of the form $(r, \emptyset)$ is called a *material resource*. It coincides with the plain resource similarity, i.e.

$$r \approx s \quad \Leftrightarrow \quad (r, \emptyset) \approx (s, \emptyset).$$

- A generalized resource of the form $({}^\bullet\alpha, \alpha)$ is an *activity resource*. For $\alpha, \beta \in \mathcal{M}(T) : ({}^\bullet\alpha, \alpha) \approx ({}^\bullet\beta, \beta) \quad \Leftrightarrow \quad (\alpha^\bullet, \emptyset) \approx (\beta^\bullet, \emptyset)$, i.e. activity steps can be interchanged, iff they produce equivalent material resources.

Generalized Resource Similarity: Properties

Generalized resource similarity is undecidable ☹, since the plain resource similarity is undecidable.
Generalized resource similarity is closed under addition of pairs of resources, i.e.

$$(r, \alpha) \approx (s, \beta) \,\&\, (u, \gamma) \approx (v, \delta) \;\Rightarrow\; (r + u, \alpha + \gamma) \approx (s + v, \beta + \delta).$$

Hence, generalized similarity has a finite AT-basis.

Generalized Resource Similarity is Finitely-Based

To construct the ground AT-basis for the generalized resource similarity we follow the same scheme, and define a partial order $\sqsubseteq$ (abuse of notation) on pairs of generalized resources as follows:

$$\Big((r,\alpha),(s,\beta)\Big) \sqsubseteq \Big((u,\gamma),(v,\delta)\Big) \overset{def}{\Leftrightarrow} (r,s) \sqsubseteq (u,v) \;\&\; (\alpha,\beta) \sqsubseteq (\gamma,\delta).$$

Then check additionally that each generalized resource (r,α) in these minimal elements meets the requirement ${}^\bullet\alpha \subseteq r$.

Theorem 4

Let $R(N)$ be the generalized resource similarity for a labeled Petri netN. Then $R(N) = R(N)^{AT}$, the ground basis $R_s(N)$ is an AT-basis of $R(N)$, and $R_s(N)$ is finite.

Generalized Resource Similarity: Open Problem 3

Special kinds of generalized resource similarity on material, or on activity resources, can be reduced to plain resource similarity.

To find such pairs of similar generalized resources computing approximations of resource bisimulation can be used.

OPEN PROBLEM

Is it possible to extend the notion of resource bisimulation and the related theory to generalized resource similarity?

Conclusion

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- Resource equivalences:
  1. Resource similarity
  2. Resource bisimulation
  3. Conditional resource similarity
  4. Generalized resource similarity
- All are finitely-based.
- All except (2) are undecidable. It's an open problem, whether (2) is decidable.
- Approximation of (2) can be effectively computed. It is an open problem to compute the size of its AT-basis.
- Approximation of (2) can be used for computing subsets of (1) and (3), and of special kinds of (4). It is an open question, whether the theory can be extended to (4).

Further Research

- Week resource similarity/bisimulation.

  Vladimir A. Bashkin. On the Resource Equivalences in Petri nets with Invisible Transitions // PNSE'2017

- Petri net reduction based on resource similarity.
- Implementation and tools.
- ...

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